Boundary behavior of a constrained Brownian motion between reflecting-repellent walls

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Abstract

Stochastic variational inequalities provide a unified treatment for stochastic differential equations living in a closed domain with normal reflection and (or) singular repellent drift. When the domain is a polyhedron, we prove that the reflected-repelled Brownian motion does not hit the non-smooth part of the boundary. A sufficient condition for non-hitting a face of the polyhedron is derived from the one-dimensional case. A complete answer to the question of attainability of the walls of the Weyl chamber may be given for a radial Dunkl process.

1 Introduction

There have been many works about stochastic differential equations with reflection on the boundary of a domain. In some of them the domain is a convex polyhedron ([17], [29], [30], [11], [12]). A typical question in this setting is the following: does the continuous process hit the non-smooth part of the boundary? The answer depends on the drift and diffusion coefficients of the process and on the direction of reflection (normal or oblique). In particular, R.Williams [30] has proven that the Brownian motion with a skew symmetry condition on the direction of reflection does not touch the intersections of the faces of the polyhedron.

On the other hand there also exists an extensive literature about non-colliding Brownian particles ([15], [3], [18], [16], [24]). Most of these works originate in the study of the eigenvalues of Gaussian matrix processes. These eigenvalues are solutions to systems of stochastic differential equations with a singular drift that prevents the particles from colliding. Extensions of these systems are Dunkl processes [25] that have recently been developed in connection with harmonic analysis on symmetric spaces. The radial part of a Dunkl process may be considered as a Brownian motion perturbed by a singular drift which forces the process to live in a cone generated by the intersection of a finite set of half-spaces ([9], [10]). Depending on the values of some parameter, the process may touch the walls of the cone or not.

Actually it is possible to unify both theories of (normal) reflection and strong repulsion within a common framework. This is the role of stochastic variational inequalities, also called multivalued stochastic differential equations (MSDE) that were mainly developed by E.Cépa ([4], [5]). These equations are associated to a convex function in a domain of $\mathbb{R}^d$. Depending on the boundary behavior of this function the diffusion will (normally) reflect on the boundary, hit the boundary without local time, or live in the open domain. We shall here follow this way and concentrate on a Brownian motion living in a convex polyhedral domain, bounded or unbounded. To each face of the polyhedron is associated a repelling force with normal reflection when the repulsion is not strong enough. In this setting we shall

ask whether the process may hit the various faces. Our first task will be to rule out the possibility of hitting the intersection of two faces. Once this is achieved, the problem is now basically one-dimensional and we may use the ordinary scale function of real diffusions.

In several previous works ([20], [8]), this issue has been studied in the particular case of the hyperplanes $H_{ij} := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_i = x_j\}$, $i \neq j$ and presented as the problem of collisions between Brownian particles. There is a simple collision if two coordinates coincide and a multiple collision if at least three coordinates coincide at the same time. Because the d-dimensional Brownian motion does not hit the intersection of two hyperplanes, one can guess that an additional drift does not change anything. However a rigorous proof is necessary because the singularity of the drift makes useless the usual Girsanov change of probability measure. The counterexample of Bass and Pardoux [1] also showed that uniform nondegeneracy of the diffusion term does not preclude multiple collisions.

As in [8] where the particular case of electrostatic repulsion was considered, our proof only uses basic tools from stochastic calculus, mainly McKean's martingale method [22] which was already used in [2] to prove non-collision for the eigenvalues of Wishart processes. Another way could be to use the theory of Dirichlet forms as done in [20] where a general condition of non-collision has been obtained.

The paper is organized as follows. In Section 2 we introduce basic definitions and notations. The main features about stochastic variational inequalities are also recalled. Section 3 is devoted to non attainability of the edges of the polyhedron. In Section 4 we give a sufficient condition of non attainability of a single face. Section 5 presents some applications to Brownian particles with nearest neighbor interaction, Wishart processes and Dunkl processes.

2 Multivalued stochastic differential equation in a polyhedral domain

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ be a filtered probability space endowed with the usual conditions and $B = (B_t)$ be a $(\mathcal{F}_t)$-adapted d-dimensional Brownian motion starting from the origin. Let

$$\Phi : \mathbb{R}^d \to (-\infty, +\infty] \tag{1}$$

be a lower semi-continuous convex function such that

$$dom(\Phi) := \{x : \Phi(x) < \infty\} \tag{2}$$

has nonempty interior. Let

$$D := Int(dom(\Phi)) \ . \tag{3}$$

For simplicity of notation, we will assume that Φ is C^1 on D. If $x \in \partial D$, we say that the unit vector $n(x)$ is a unit inward normal to D at x if

$$n(x).(x - z) \leq 0 \tag{4}$$

for any $z \in \overline{D}$. Based on the results in [4], the following theorem has been proved in [6] (see also Theorem 2.2 in [7]).

Theorem 1 *For any $\mathcal{F}_0$-measurable random variable X_0 with values in $\overline{D}$, there exist a unique continuous $(\mathcal{F}_t)$-adapted process $X = \{X_t, 0 \leq t < \infty\}$ with values in $\overline{D}$ and a unique continuous $(\mathcal{F}_t)$-adapted non-decreasing process $L = \{L_t, 0 \leq t < \infty\}$ such that*

$$\begin{array}{lcll} X_t & = & X_0 + B_t - \int_0^t \nabla\Phi(X_s)\, ds + \int_0^t n_s\, dL_s & \qquad 0 \leq t < \infty \\ L_t & = & \int_0^t \mathbf{1}_{\{X_s \in \partial D\}} dL_s & \qquad 0 \leq t < \infty \end{array} \tag{5}$$

where n_s is dL_s-a.e. a unit inward normal to D at X_s. For any $0 < T < \infty$,

$$\int_0^T \mathbf{1}_{\{X_s \in \partial D\}} ds \;=\; 0 \tag{6}$$

and

$$\int_0^T |\nabla\Phi(X_s)|\, ds \;<\; \infty\;. \tag{7}$$

From now on we concentrate on a particular polyhedral setting. Let $\mathbf{I} := \{1, \ldots, m\}$ where $m \geq 1$. We consider a convex function Φ of the following form

$$\Phi(x) \;:=\; \sum_{i\in\mathbf{I}} \phi_i(x.n_i - a_i) \tag{8}$$

where for any $i \in \mathbf{I}$,

$$\begin{array}{ll} \phi_i & \text{is a convex l.s.c. function },\;\; \phi_i = +\infty \;\text{ on } (-\infty,0),\;\; \phi_i \text{ is } C^1 \text{ on } (0,+\infty) \\ n_i & \text{is a unit vector} \\ a_i & \text{is a real number .} \end{array} \tag{9}$$

We may assume all n_i are different. Then,

$$\begin{array}{lcl} \nabla\Phi(x) & = & \sum_{i\in\mathbf{I}} n_i\, \phi_i'(x.n_i - a_i) \\ D & = & \{x \in \mathbb{R}^d : \, x.n_i \,>\, a_i \;\; \forall i \in \mathbf{I}\} \\ \overline{D} & = & \{x \in \mathbb{R}^d : \, x.n_i \,\geq\, a_i \;\; \forall i \in \mathbf{I}\}\;. \end{array} \tag{10}$$

As D is not empty, there exists a ball with center $y \in D$ and radius $b > 0$ included in D. Let X_t be the solution given by Theorem 1. For $i \in \mathbf{I}$ let

$$U_t^i := X_t.n_i \;-\; a_i\;. \tag{11}$$

We will need a strengthening of inequality (7) ([7],Th.2.2).

Lemma 2 *For any $i \in I$, for any $0 < t < \infty$,*

$$\int_0^t |\phi_i'(U_s^i)|\, ds \;<\; \infty\;. \tag{12}$$

Proof. This is clear if $\phi_i'(0+) > -\infty$. Let

$$\mathbf{J} := \{j \in \mathbf{I} : \phi_j'(0+) = -\infty\} \tag{13}$$

and let $0 < \varepsilon < b$ be such that $\phi_j'(u) < 0$ for any $j \in \mathbf{J}$ and $u \in (0, \varepsilon)$. For $\mathbf{K} \subset \mathbf{J}$ let

$$A_{\mathbf{K}} := \{x \in \overline{D} : x.n_j < a_j + \varepsilon \;\; \forall j \in \mathbf{K}, \quad x.n_j \geq a_j + \varepsilon \;\; \forall j \in \mathbf{J} \setminus \mathbf{K}\}\;. \tag{14}$$

Then for $t > 0$

$$\int_0^t \mathbf{1}_{A_{\mathbf{K}}}(X_s)|\sum_{j\notin\mathbf{K}} n_j\, \phi_j'(U_s^j)|ds \leq \sum_{j\notin\mathbf{K}} \int_0^t \mathbf{1}_{A_{\mathbf{K}}}(X_s)|\phi_j'(U_s^j)|ds \;<\; \infty\;. \tag{15}$$

Using (7) we get

$$\int_0^t \mathbf{1}_{A_{\mathbf{K}}}(X_s) | \sum_{j\in\mathbf{K}} n_j\, \phi_j'(U_s^j)|ds \;<\; \infty \tag{16}$$

and therefore

$$\begin{array}{rcl} -(b-\varepsilon)\sum_{j\in\mathbf{K}}\int_0^t \mathbf{1}_{A_{\mathbf{K}}}(X_s)\phi_j'(U_s^j)\,ds & \leq & \int_0^t \mathbf{1}_{A_{\mathbf{K}}}(X_s)\sum_{j\in\mathbf{K}}(y-X_s).n_j|\phi_j'(U_s^j)|ds \\ & \leq & \int_0^t \mathbf{1}_{A_{\mathbf{K}}}(X_s)|y-X_s|\sum_{j\in\mathbf{K}}|\phi_j'(U_s^j)|ds \\ & < & \infty \end{array} \tag{17}$$

from the continuity of X on $[0,t]$. Then for any $j\in\mathbf{J}$

$$\begin{array}{rcl} \int_0^t |\phi_j'(U_s^j)|\,ds & = & \int_0^t \mathbf{1}_{\{U_s^j<\varepsilon\}}|\phi_j'(U_s^j)|\,ds \,+\, \int_0^t \mathbf{1}_{\{U_s^j\geq\varepsilon\}}|\phi_j'(U_s^j)|\,ds \\ & = & \sum_{j\in\mathbf{K}\subset\mathbf{J}}\int_0^t \mathbf{1}_{A_{\mathbf{K}}}(X_s)\,|\phi_j'(U_s^j)|ds \,+\, \int_0^t \mathbf{1}_{\{U_s^j\geq\varepsilon\}}|\phi_j'(U_s^j)|\,ds \\ & < & \infty\,. \end{array} \tag{18}$$

□

For any $\mathbf{J}\subset\mathbf{I}$, $\mathbf{J}\neq\emptyset$, we set

$$\begin{array}{rcl} H_{\mathbf{J}} & := & \{x\in\mathbb{R}^d : x.n_j = a_j \;\;\forall j\in\mathbf{J}\} \\ K_{\mathbf{J}} & := & \{x\in\mathbb{R}^d : x.n_j = a_j \;\;\forall j\in\mathbf{J},\;\; x.n_j > a_j \;\;\forall j\notin\mathbf{J}\} \\ \sigma_{\mathbf{J}} & := & \inf\{t>0\,:\,X_t\,\in H_{\mathbf{J}}\} \\ \tau_{\mathbf{J}} & := & \inf\{t>0\,:\,X_t\,\in K_{\mathbf{J}}\}\;\;. \end{array} \tag{19}$$

Lemma 3 *Let $\mathbf{J}\subset\mathbf{I}$ and $V:=span\{n_j, j\in\mathbf{J}\}$. If $n(x)$ is a unit inward normal to D at $x\in K_{\mathbf{J}}$, then $n(x)\in V$.*

Proof. Let $v\perp V$. For $\varepsilon>0$ small enough,

$$z_1 = x+\varepsilon v \qquad z_2 = x-\varepsilon v$$

satisfy

$$\begin{array}{ll} z_1.n_j = a_j \;\;\forall j\in\mathbf{J} & \qquad z_1.n_i > a_i \;\;\forall i\notin\mathbf{J} \\ z_2.n_j = a_j \;\;\forall j\in\mathbf{J} & \qquad z_2.n_i > a_i \;\;\forall i\notin\mathbf{J}\,. \end{array}$$

Then

$$n(x).(x-z_1)\leq 0 \qquad n(x).(x-z_2)\leq 0$$

and therefore

$$n(x).v = 0\,.$$

□

3 Nonattainability of the edges

This section is devoted to the proof of the following theorem.

Theorem 4 *For any $\mathbf{J}\subset\mathbf{I}$ with $card(\mathbf{J})\geq 2$,*

$$\mathbb{P}(\sigma_{\mathbf{J}}=\infty)\;=\;1\,.$$

Proof. a/ We first consider the initial condition X_0. From (6) we deduce that for any $u > 0$ there exists $0 < v < u$ such that $X_v \in D$ a.s. Using the continuity of paths and the Markov property we may and do assume that $X_0 \in D$ in order to prove that $\sigma_{\mathbf{J}} = \infty$ a.s.

b/ We will also assume that

$$\max_{i\in\mathbf{I}} \phi_i'(0+) \; < \; 0 \; . \tag{20}$$

If not we introduce for any $0 < T < \infty$ the equivalent probability measure $\mathbb{Q}$ defined on $\mathcal{F}_T$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \exp\{c(B_T.\sum_{i\in\mathbf{I}} n_i) - \frac{1}{2}c^2T|\sum_{i\in\mathbf{I}} n_i|^2\}$$

where

$$c > \max_{i\in\mathbf{I}} \phi_i'(0+) \; .$$

The continuous process

$$B_t' := B_t - ct\sum_{i\in\mathbf{I}} n_i$$

is a $\mathbb{Q}$-Brownian motion on $[0, T]$ and now

$$dX_t = dB_t' - \sum_{i\in\mathbf{I}} n_i\psi_i'(X_t.n_i - a_i)dt - n_t dL_t$$

where

$$\psi_i(u) := \phi_i(u) - cu \quad i \in \mathbf{I} \; .$$

If $\mathbb{Q}(\sigma_{\mathbf{J}} < T) = 0$ then $\mathbb{P}(\sigma_{\mathbf{J}} < T) = 0$ and if this is true for any T we obtain $\mathbb{P}(\sigma_{\mathbf{J}} = \infty) = 1$.

c/ We are now going to prove that $\sigma_{\mathbf{I}} = \tau_{\mathbf{I}} = \infty$ a.s. (with $m \geq 2$). For any $\mathbf{J} \subset \mathbf{I}$ let

$$\begin{array}{lcl} V_{\mathbf{J}} & := & span\{n_j, j \in \mathbf{J}\} \\ q_{\mathbf{J}} & := & dim\, V_{\mathbf{J}} \\ \pi_{\mathbf{J}} & := & \text{orthogonal projection onto } \; V_{\mathbf{J}} \; . \end{array} \tag{21}$$

If $q_{\mathbf{I}} = 1$, then $m = 2$, $n_1 + n_2 = 0$ and $H_{\mathbf{I}} = K_{\mathbf{I}} = \emptyset$. Assume now $q_{\mathbf{I}} \geq 2$ and $H_{\mathbf{I}} \neq \emptyset$. Choose some $z \in H_{\mathbf{I}}$ and set

$$Z_t := \pi_{\mathbf{I}}(X_t - z) \; . \tag{22}$$

Then

$$Z_t \; = \; Z_0 \; + \; C_t \; - \sum_{i\in\mathbf{I}} \int_0^t n_i\, \phi_i'(U_s^i)ds \; + \; \int_0^t n_s\, dL_s \tag{23}$$

where C is a $q_{\mathbf{I}}$-dimensional Brownian motion. Set

$$S_t \; := \; |Z_t|^2 \; .$$

Then

$$S_t = S_0 + 2\int_0^t Z_s.dC_s - 2\sum_{i\in\mathbf{I}} \int_0^t U_s^i\, \phi_i'(U_s^i)ds + 2\int_0^t Z_s.n_s\, dL_s + q_{\mathbf{I}}t. \tag{24}$$

From Lemma 2 we deduce that on $\partial D = \cup_{\mathbf{J}\subset\mathbf{I}} K_{\mathbf{J}}$

$$Z_s.n_s \; = \; (X_s - z).n_s \; = \; 0$$

and thus

$$\int_0^t Z_s.n_s\, dL_s = 0\ .$$

Let $0 < T < \infty$. For $t < \tau_{\mathbf{I}} \wedge T$,

$$\log S_t \;=\; \log S_0 \;+\; 2\int_0^t \frac{Z_s.dC_s}{S_s} - 2\sum_{i\in\mathbf{I}} \int_0^t \frac{U_s^i\,\phi_i'(U_s^i)}{S_s} ds \;+\; (q_{\mathbf{I}}-2)\int_0^t \frac{ds}{S_s}\ . \tag{25}$$

From the assumption made in b/ there exists $0 < c \le \infty$ such that $\phi_i' \le 0$ on $(0,c]$ and

$$\begin{array}{rcl} -\int_0^t \frac{U_s^i\,\phi_i'(U_s^i)}{S_s} ds & \ge & -\int_0^t \frac{U_s^i\,\phi_i'(U_s^i)}{S_s}\,\mathbf{1}_{\{U_s^i \ge c\}} ds \\ & \ge & -\frac{1}{c}\int_0^T |\phi_i'(U_s^i)|\, ds \\ & > & -\infty\ . \end{array} \tag{26}$$

We now proceed as in ([22],p.47). As $t \to \tau_{\mathbf{I}} \wedge T$, the local martingale part in the r.h.s. of (25) either converges to a finite limit or oscillates between $+\infty$ and $-\infty$. Thus it does not converge to $-\infty$ and a.s. $S_{\tau_{\mathbf{I}}\wedge T} > 0$. Therefore

$$\mathbb{P}(\tau_{\mathbf{I}} \le T) = 0$$

and the conclusion follows since T is arbitrary.

d/ Let now $\mathbf{J} \subset \mathbf{I}$ with $2 \le |\mathbf{J}| \le m-1$. We shall show by a backward induction on $|\mathbf{J}|$ that $\mathbb{P}(\tau_{\mathbf{J}} = \infty) = 1$. Remark that the backward induction assumption entails the equality $\sigma_{\mathbf{J}} = \tau_{\mathbf{J}}$ a.s.. As previously done we may assume $q_{\mathbf{J}} \ge 2$ and $K_{\mathbf{J}} \ne \emptyset$. Select now $z \in K_{\mathbf{J}}$ and set

$$\begin{array}{rcl} Z_t & := & \pi_{\mathbf{J}}(X_t - z) \\ & = & Z_0 \,+\, C_t \,-\sum_{j\in\mathbf{J}}\int_0^t n_j\,\phi_j'(U_s^j)ds - \sum_{i\notin\mathbf{J}}\int_0^t \pi_{\mathbf{J}} n_i\,\phi_i'(U_s^i)ds + \int_0^t \pi_{\mathbf{J}} n_s\, dL_s \end{array} \tag{27}$$

where C is a $q_{\mathbf{J}}$-dimensional Brownian motion. Let again $S_t := |Z_t|^2$. For $\varepsilon > 0$ and $r > 0$ we set

$$\begin{array}{rcl} \tau_\varepsilon & := & \inf\{t > 0\,:\, S_t + \min_{i\notin\mathbf{J}}(U_t^i)^2 \le 2\,\varepsilon^2\} \\ \rho_r & = & \inf\{t > 0\,:\, |X_t| \ge r\}\ . \end{array} \tag{28}$$

From the induction assumption we infer that $\tau_\varepsilon \to \infty$ as ε goes to 0. Let $0 < T < \infty$. We introduce the equivalent probability measure $\mathbb{Q}$ defined on $\mathcal{F}_T$ by

$$\begin{array}{rcl} \frac{d\mathbb{Q}}{d\mathbb{P}} & = & \exp\{\int_0^{\tau_\varepsilon\wedge\rho_r\wedge T} \sum_{i\notin\mathbf{J}} \mathbf{1}_{\{U_s^i \ge \varepsilon\}} \phi_i'(U_s^i)\, n_i.dC_s \\ & & \quad -\frac{1}{2}\int_0^{\tau_\varepsilon\wedge\rho_r\wedge T} |\sum_{i\notin\mathbf{J}} \mathbf{1}_{\{U_s^i\ge\varepsilon\}}\phi_i'(U_s^i)\,\pi_{\mathbf{J}} n_i|^2\, ds\}\ . \end{array} \tag{29}$$

Then

$$D_t \;:=\; C_t - \int_0^{\tau_\varepsilon\wedge\rho_r\wedge T} \sum_{i\notin\mathbf{J}} \mathbf{1}_{\{U_s^i\ge\varepsilon\}}\phi_i'(U_s^i)\pi_{\mathbf{J}} n_i\, ds$$

is a $q_{\mathbf{J}}$-dimensional $\mathbb{Q}$-Brownian motion on $[0,T]$. For $t \le \tau_\varepsilon \wedge \rho_r \wedge T$,

$$\begin{array}{rcl} S_t & = & S_0 + 2\int_0^t Z_s.dD_s - 2\sum_{i\in\mathbf{I}}\int_0^t U_s^i\,\phi_i'(U_s^i)ds \;-\; 2\sum_{i\notin\mathbf{J}}\int_0^t \mathbf{1}_{\{U_s^i<\varepsilon\}}\, Z_s.n_i\,\phi_i'(U_s^i)\, ds \\ & & +2\sum_{\mathbf{L}\subset\mathbf{I},\mathbf{L}\not\subset\mathbf{J}}\int_0^t \mathbf{1}_{K_{\mathbf{L}}}(X_s)\, Z_s.n_s dL_s + q_{\mathbf{J}} t \end{array} \tag{30}$$

and for $t < \sigma_{\mathbf{J}} \wedge \tau_\varepsilon \wedge \rho_r \wedge T$,

$$\begin{array}{rcl}\log S_t & = & \log S_0 + 2\int_0^t \frac{Z_s.dD_s}{S_s} - 2\sum_{j\in\mathbf{J}} \int_0^t \frac{U_s^j\,\phi_j'(U_s^j)}{S_s} ds \\ & & -2\sum_{i\notin\mathbf{J}} \int_0^t \mathbf{1}_{\{U_s^i<\varepsilon\}} \frac{\phi_i'(U_s^i)}{S_s} Z_s.n_i\, ds \\ & & +2\sum_{\mathbf{L}\subset\mathbf{I},\mathbf{L}\not\subset\mathbf{J}} \int_0^t \mathbf{1}_{K_{\mathbf{L}}}(X_s) \frac{Z_s.n_s}{S_s} dL_s \\ & & +(q_{\mathbf{J}}-2)\int_0^t \frac{ds}{S_s}\ .\end{array} \tag{31}$$

From the induction hypothesis and the continuity of paths, if $\sigma_{\mathbf{J}} < \infty$ for any $\mathbf{L} \not\subset \mathbf{J}$ there exists an interval $(\sigma_{\mathbf{J}} - \delta, \sigma_{\mathbf{J}}]$ of positive length on which $X_s \notin K_{\mathbf{L}}$. Therefore

$$-\int_0^{\sigma_{\mathbf{J}}\wedge\tau_\varepsilon\wedge\rho_r\wedge T} \mathbf{1}_{K_{\mathbf{L}}}(X_s)\frac{Z_s.n_s}{S_s} dL_s > -\infty\ . \tag{32}$$

For $s < \tau_\varepsilon$, if $U_s^i < \varepsilon$ for some $i \notin \mathbf{J}$, then $S_s \geq \varepsilon^2$ and we obtain as well

$$-\int_0^{\sigma_{\mathbf{J}}\wedge\tau_\varepsilon\wedge\rho_r\wedge T} \mathbf{1}_{\{U_s^i<\varepsilon\}}\frac{\phi_i'(U_s^i)}{S_s} Z_s.n_i\, ds > -\infty\ . \tag{33}$$

The other terms behave as in c/ and thus

$$0 = \mathbb{Q}(\sigma_{\mathbf{J}} \leq \tau_\varepsilon \wedge \rho_r \wedge T) = \mathbb{P}(\sigma_{\mathbf{J}} \leq \tau_\varepsilon \wedge \rho_r \wedge T)\ . \tag{34}$$

Letting ε go to 0, r and T to ∞ we get

$$\mathbb{P}(\sigma_{\mathbf{J}} = \infty) = 1$$

and we are done. □

4 Keeping off from a wall

We first recall some facts in the one-dimensional setting [21]. Let $\phi : \mathbb{R} \to (-\infty, +\infty]$ be a convex lower semicontinuous function. Assume $\phi = +\infty$ on $(-\infty, 0)$ and C^1 on $(0, +\infty)$. Consider the one-dimensional equation

$$\begin{array}{rcl} dY_t & = & dB_t - \phi'(Y_t)dt + \frac{1}{2}dL_t^0 \\ Y_t & \geq & 0 \end{array} \tag{35}$$

where L^0 is the local time of Y at 0. There are three types of boundary behavior:

	repulsion
$\phi(0) < \infty$	weak: local time not zero
$\phi(0) = \infty$, $\int_{0+} \exp\{2\phi\} < \infty$	middle: local time zero
$\phi(0) = \infty$, $\int_{0+} \exp\{2\phi\} = \infty$	strong: boundary not hit

We shall check the behavior of the multidimensional process X accords with this classification in the neighborhood of the faces of the polyhedron. For any $i \in \mathbf{I}$ we respectively write $H_i, K_i, \sigma_i, \tau_i$ in place of $H_{\{i\}}, K_{\{i\}}, \sigma_{\{i\}}, \tau_{\{i\}}$.

Proposition 5 *For any $i \in \mathbf{I}$ such that $\phi_i(0) = \infty$ and any $t > 0$,*

$$\int_0^t \mathbf{1}_{H_i}(X_s)\, dL_s\ =\ 0\ . \tag{36}$$

Proof. From the occupation times formula and Lemma 1 we obtain

$$\int_0^\infty L_t^a(U_i)\,|\phi_i'(a)|\,da = \int_0^t |\phi_i'(U_s^i)|\,ds < \infty \tag{37}$$

and from $\phi_i(0) = \infty$ and the continuity of $a \mapsto L_t^a(U_i)$ we deduce

$$L_t^0(U_i) = 0\ . \tag{38}$$

Thus

$$\begin{array}{rcl} 0 &=& U_t^i - (U_t^i)^+ \\ &=& \int_0^t \mathbf{1}_{H_i}(X_s) n_i.dB_s - \int_0^t \mathbf{1}_{H_i}(X_s) \sum_{j\in\mathbf{I}} \phi_j'(U_s^j)\, n_i.n_j\, ds + \int_0^t \mathbf{1}_{H_i}(X_s)\, n_i.n_s\, dL_s \\ &=& \int_0^t \mathbf{1}_{K_i}(X_s)\, n_i.n_s\, dL_s \\ &=& \int_0^t \mathbf{1}_{K_i}(X_s)\, dL_s \\ &=& \int_0^t \mathbf{1}_{H_i}(X_s)\, dL_s\ . \end{array} \tag{39}$$

□

We now set for any $i \in \mathbf{I}$ and $x \geq 0$

$$p_i(x) := \int_1^x \exp\{2(\phi_i(u) - \phi_i(1))\}\, du\ .$$

Theorem 6 *For any $i \in \mathbf{I}$ such that $p_i(0) = -\infty$ or equivalently*

$$\int_{0+} \exp\{2\phi_i\} = \infty\ , \tag{40}$$

then $\mathbb{P}(\sigma_i = \infty) = \mathbb{P}(\tau_i = \infty) = 1$.

Proof. From Ito formula and Proposition 5 we obtain

$$p_i(U_t^i) = p_i(U_0^i) + \int_0^t p_i'(U_s^i)[dC_s^i - \sum_{j\neq i} n_i.n_j\, \phi_j'(U_s^j) ds + \sum_{j\neq i} \mathbf{1}_{K_j}(X_s)\, n_i.n_j\, dL_s] \tag{41}$$

where $C^i = B.n_i$ is a one-dimensional Brownian motion. As in the proof of Theorem 2, let

$$\begin{array}{rcl} \tau_\varepsilon &:=& \inf\{t > 0 \,:\, U_t^i + \min_{j\neq i}(U_t^j) \leq 2\,\varepsilon\} \\ \rho_r &=& \inf\{t > 0 \,:\, |X_t| \geq r\}\ . \end{array} \tag{42}$$

Let $0 < T < \infty$. We again introduce the equivalent probability measure $\mathbb{Q}$ defined on $\mathcal{F}_T$ by

$$\begin{array}{rcl} \frac{d\mathbb{Q}}{d\mathbb{P}} &=& \exp\{\int_0^{\tau_\varepsilon\wedge\rho_r\wedge T} \sum_{j\neq i} \mathbf{1}_{\{U_s^j \geq \varepsilon\}} \phi_j'(U_s^j)\, n_i.n_j\, dC_s^i \\ && -\frac{1}{2}\int_0^{\tau_\varepsilon\wedge\rho_r\wedge T} |\sum_{j\neq i} \mathbf{1}_{\{U_s^j \geq \varepsilon\}} \phi_j'(U_s^j)\, n_i.n_j|^2\, ds\}\ . \end{array} \tag{43}$$

Then

$$D_t^i := C_t^i - \int_0^{t\wedge\tau_\varepsilon\wedge\rho_r} \sum_{j\neq i} \mathbf{1}_{\{U_s^j \geq \varepsilon\}} \phi_j'(U_s^j)\, n_i.n_j\, ds \tag{44}$$

is a $\mathbb{Q}$-Brownian motion on $[0, T]$ and for $t \leq \tau_\varepsilon \wedge \rho_r \wedge T$,

$$p_i(U_t^i) = p_i(U_0^i) + \int_0^t p_i'(U_s^i)[dD_s^i - \sum_{j\neq i} \mathbf{1}_{\{U_s^i < \varepsilon\}} n_i.n_j\, \phi_j'(U_s^j) ds + \sum_{j\neq i} \mathbf{1}_{K_j}(X_s)\, n_i.n_j\, dL_s]\ . \tag{45}$$

As in the proof of Theorem 2, for any $j \neq i$,

$$- \int_0^{\sigma_i \wedge \tau_\varepsilon \wedge \rho_r \wedge T} \mathbf{1}_{\{U_s^i < \varepsilon\}} \, p_i'(U_s^i) \, n_i . n_j \, \phi_j'(U_s^j) \, ds \; > -\infty \tag{46}$$

and

$$+ \int_0^{\sigma_i \wedge \tau_\varepsilon \wedge \rho_r \wedge T} \mathbf{1}_{K_j}(X_s) \, p_i'(U_s^i) \, n_i . n_j \, dL_s \; > -\infty \tag{47}$$

and then

$$0 \;=\; \mathbb{Q}(\sigma_i \leq \tau_\varepsilon \wedge \rho_r \wedge T) \;=\; \mathbb{P}(\sigma_i \leq \tau_\varepsilon \wedge \rho_r \wedge T) \tag{48}$$

meaning that $\mathbb{P}(\sigma_i = \infty) = 1$. □

5 Applications

5.1 Brownian particles with nearest neighbor repulsion

H.Rost and M.E.Vares [26] have considered the following system:

$$\begin{array}{rcll} dX_t^1 & = & dB_t^1 \,+\, \phi'(X_t^2 - X_t^1)\, dt & \\ dX_t^i & = & dB_t^i \,+\, (\phi'(X_t^{i+1} - X_t^i) - \phi'(X_t^i - X_t^{i-1}))\, dt & \quad i = 2, \ldots, n-1 \\ dX_t^n & = & dB_t^n \,-\, \phi'(X_t^n - X_t^{n-1})\, dt & \end{array} \tag{49}$$

where $X_t^1 < \ldots < X_t^n$ and ϕ is a positive convex function on $(0, \infty)$ satisfying

$$\phi(0) = \infty\,, \qquad \phi(\infty) = 0\,, \qquad \int_0^1 (\phi'(x))^2 e^{-2\phi(x)}\, dx \;<\; \infty\,. \tag{50}$$

This is a MSDE where function Φ is given by (8) with $\phi_i(x) = \phi(\sqrt{2}\,x)$, $n_i = \frac{1}{\sqrt{2}}(e_{i+1} - e_i)$, $a_i = 0$ for $i = 1, \ldots, n-1$ and e_j the j-th basis vector. Condition (50) for non-collision is stronger than (40) as can be seen from Schwarz inequality:

$$\infty \;=\; (\phi(0) - \phi(1))^2 \;\leq\; \int_0^1 (\phi')^2\, e^{-2\phi} \int_0^1 e^{2\phi}\,.$$

5.2 Wishart and Laguerre processes

Wishart processes have been introduced in [2] and [3]. If B is a $n \times n$ Brownian matrix, a Wishart process with parameters n and $\delta \geq n+1$ may be obtained as a solution to the matrix-valued SDE

$$dS_t \;=\; \sqrt{S_t}\, dB_t \,+\, dB_t' \sqrt{S_t} \,+\, \delta\, I_n\, dt\,. \tag{51}$$

The eigenvalues process $(\lambda_t^1, \ldots, \lambda_t^n)$ of $\{S_t\}$ satisfies

$$d\lambda_t^i \;=\; 2\sqrt{\lambda_t^i}\, dW_t^i \,+\, (\delta \,+\, \sum_{j \neq i} \frac{\lambda_t^i + \lambda_t^j}{\lambda_t^i - \lambda_t^j})\, dt \qquad 1 \leq i \leq n\,, \tag{52}$$

and the square roots $r_t^i = \sqrt{\lambda_t^i}$

$$dr_t^i \;=\; dW_t^i \,+\, \frac{1}{2} \frac{\delta - n}{r_t^i} dt \,+\, \frac{1}{2} \sum_{j \neq i} (\frac{1}{r_t^i + r_t^j} \,+\, \frac{1}{r_t^i - r_t^j})\, dt \tag{53}$$

where $(W^i, \dots, W^n)$ is a n-dimensional Brownian motion. N.Demni [14] has remarked that this system is a MSDE with

$$\Phi(r^1, \dots, r^n) \;=\; -\frac{1}{2}[(\delta - n)\sum_i \log\, r^i \,+\, \sum_{i>j} \log(r^i + r^j) \,+\, \sum_{i>j} \log(r^i - r^j)] \tag{54}$$

on $\{0 < r^1 < \ldots < r^n\}$ and ∞ elsewhere. The system (53) has a strong solution for $\delta > n$. If $\delta = n$, we must add to the right hand side of (53) a local time at 0 that disappears in (52). It has been proven in [3] that the eigenvalues never collide and if moreover $\delta \geq n+1$ the smallest one never vanishes. This is in accordance with Theorem 6.

Laguerre processes are Hermitian versions of Wishart processes. Only constants are changed in (52), (53) and (54).

5.3 Reflection groups and Dunkl processes

We only give a short introduction to this topic and refer to [19] and [25] for more details. For $\alpha \in \mathbb{R}^N \setminus \{0\}$ we denote by s_α the orthogonal reflection with respect to the hyperplane H_α perpendicular to α:

$$s_\alpha(x) \;=\; x \,-\, 2\frac{\alpha.x}{|\alpha|^2}\;. \tag{55}$$

A finite subset $R \subset \mathbb{R}^N \setminus \{0\}$ is called a *root system* if for all $\alpha \in R$

$$\begin{aligned} R \cap \mathbb{R}\alpha \;&=\; \{\alpha, -\alpha\}\;; \\ s_\alpha(R) \;&=\; R\,. \end{aligned} \tag{56}$$

The group $W \subseteq O(N)$ which is generated by the reflections $\{s_\alpha, \alpha \in R\}$ is called the *reflection group* associated with R. Each hyperplane $H_\beta := \{x \in \mathbb{R}^N : \beta.x = 0\}$ with $\beta \in \mathbb{R}^N \setminus \cup_{\alpha\in R} H_\alpha$ separates the root system R into R_+ and R_-. Such a set R_+ is called a *positive subsystem* and defines the *positive Weyl chamber* C by

$$C := \{x \in \mathbb{R}^N : \alpha.x > 0 \;\; \forall \alpha \in R_+\}\;. \tag{57}$$

A subset S of R_+ is called *simple* if S is a vector basis for $span(R)$. The elements of S are called *simple*. Such a subset exists, is unique and we actually get

$$C = \{x \in \mathbb{R}^N : \alpha.x > 0 \;\; \forall \alpha \in S\}\;. \tag{58}$$

A function $k : R \to \mathbb{R}$ on the root system is called a *multiplicity function* if it is invariant under the natural action of W on R. If the multiplicity function k is positive on R_+, we define the radial Dunkl process X^W as the $\overline{C}$-valued continuous path Markov process whose generator is given by

$$\mathcal{L}_k^W u(x) \;=\; \frac{1}{2}\Delta u(x) \,+\, \sum_{\alpha\in R_+} k(\alpha)\,\frac{\alpha.\nabla u(x)}{\alpha.x} \tag{59}$$

for $u \in C^2(\overline{C})$ with the boundary condition $\alpha.\nabla u(x) = 0$ for $x \in H_\alpha$. Then X^W may be viewed as the solution to the MSDE

$$dY_t \;=\; dB_t \,-\, \nabla\Phi(Y_t)\,dt$$

where B is a Brownian motion and

$$\Phi(y) \;=\; \sum_{\alpha\in R_+} k(\alpha)\, \log(\alpha.y) \tag{60}$$

on C and $\Phi = \infty$ elsewhere. It was proved in ([9], [10]) that this equation has a unique strong solution and if moreover $k(\alpha) \geq 1/2$ for any $\alpha \in R$ then the process never hits the walls H_α of the Weyl chamber. In [14], it is proved that if $k(\alpha) < 1/2$ for a simple root α, then the process hits H_α a.s. As a consequence of this result and of Theorem 6 (see also the statement at the bottom of p.117 in [10]), we are in a position to classify the boundary behavior of the radial Dunkl process in the Weyl chamber.

Proposition 7 *For any $\alpha \in R_+$ let $\sigma_\alpha := \inf\{t > 0 \,:\, X_t^W \in H_\alpha\}$.*

- *If $\alpha \in R_+ \setminus S$, then $\mathbb{P}(\sigma_\alpha = \infty) = 1$,*
- *If $\alpha \in S$ and $k(\alpha) \geq 1/2$, then $\mathbb{P}(\sigma_\alpha = \infty) = 1$,*
- *If $\alpha \in S$ and $k(\alpha) < 1/2$, then $\mathbb{P}(\sigma_\alpha < \infty) = 1$.*

5.4 Trigonometric and hyperbolic interactions

Others interactions have been studied in [7].

The trigonometric system ([15], [18], [28]) reads

$$\begin{array}{ll} dX_t^j \;=\; dB_t^j \;+\; \frac{\gamma}{2}\sum_{k\neq j} \cot \frac{X_t^j - X_t^k}{2} & \quad 1 \leq j \leq n \\ X_t^1 \leq X_t^2 \leq \ldots \leq X_t^n \leq X_t^1 \;+\; 2\pi & \end{array} \tag{61}$$

This can be interpreted as the solution to the MSDE associated with

$$\Phi(x) \;=\; \sum_{i>j} \phi(x.\frac{e_i - e_j}{\sqrt{2}}) \;+\; \sum_{i<j} \phi(x.\frac{e_i - e_j}{\sqrt{2}} \;+\; \pi\sqrt{2}) \tag{62}$$

where

$$\begin{array}{lcll} \phi(u) & = & -\gamma \,\log\left(\,\sin \frac{u}{\sqrt{2}}\right) & 0 < u < \frac{\pi}{\sqrt{2}} \\ & = & \infty & \text{elsewhere.} \end{array} \tag{63}$$

It has been proved in [7] there exist a.s. collisions if $\gamma < 1/2$.

The hyperbolic system ([23], [27]) is

$$\begin{array}{ll} dX_t^j \;=\; dB_t^j \;+\; \gamma \sum_{k\neq j} \coth\left(X_t^j - X_t^k\right) & \quad 1 \leq j \leq n \\ X_t^1 \leq X_t^2 \leq \ldots \leq X_t^n \,. & \end{array} \tag{64}$$

In this case

$$\Phi(x) \;=\; \sum_{1\leq j<k\leq n} \phi(x.\frac{e_k - e_j}{\sqrt{2}}) \tag{65}$$

with

$$\begin{array}{lcll} \phi(u) & = & -\gamma \,\log\left(\,\sinh(\sqrt{2}u)\right) & u > 0 \\ & = & \infty & \text{elsewhere.} \end{array} \tag{66}$$

and collisions occur with positive probability if $\gamma < 1/2$.

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